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Nonlinear dynamics of weak ferromagnetic spin chains

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Abstract. The nonlinear dynamics of both the isotropic and single ion anisotropic Heisenberg ferromagnetic spin chains with Dzialoshinski–Moriya-type weak anisotropic interaction has been studied in the classical continuum limit. In both cases integrable weak ferromagnetic models exhibiting soliton-like elementary spin excitations have been identified. A class of spin wave solutions have also been reported in the anisotropic case.

1. Introduction

Magnetic systems with different kinds of interactions have acted as important dynamical models exhibiting interesting nonlinear phenomena. To be specific, the quasi-one-dimensional classical continuum Heisenberg ferromagnetic spin chains with magnetic interactions such as bilinear isotropic exchange, single ion anisotropy due to the crystal field effect, inhomogeneity in the exchange interaction, interaction with an external magnetic field, etc, have been identified as integrable models with localized spin excitations such as solitons and domain walls under different circumstances [1–5]. In addition to the above common types of simple but dominant interactions there exist certain interactions which are less spoken of in the literature of nonlinear dynamics of ferromagnets due to the mathematical complexity of their representations in the Hamiltonian and in the governing dynamical equations. Important among them are the biquadratic exchange interaction [6, 7] and the weak or Dzialoshinski–Moriya (DM) anisotropic interaction [8]. Recently, a specific integrable isotropic biquadratic ferromagnetic spin chain with soliton modes has been identified [9].

Though the weak or DM interaction in magnets was identified quite some time ago [9], the revival of interest in weak ferromagnets has occurred only recently. This is because of their important role in insulators, spin glasses and in the low temperature phase of copper oxide superconductors and in phase transition studies ([10, 11] and references therein). Also very recently, attempts have been made to study Heisenberg models of quantum spin systems with DM interactions (see, e.g., [12] and references therein). However, so far the nonlinear dynamics of weak ferromagnetic spin systems has not been investigated from the integrability and soliton points of view at the classical level. Hence, in the present paper, we study at length the nonlinear dynamics of both the isotropic and single ion anisotropic one-dimensional spin chains with the weak or DM interaction. After presenting the model and deriving the equation of motion in section 2, we investigate the underlying nonlinear dynamics of the weak isotropic and anisotropic chains in sections 3 and 4 respectively. Conclusions are given in section 5.

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2. The model and equation of motion

The Heisenberg Hamiltonian for a ferromagnet involving nearest neighbour spin–spin exchange interaction, weak or DM interaction and single ion anisotropic interaction due to crystal field effect can be written as [5, 8]

$$H = -\sum_i [J\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + D\boldsymbol{m} \cdot (\boldsymbol{S}_i \wedge \boldsymbol{S}_{i+1}) - A(S_i^z)^2]. \tag{2.1}$$

In equation (2.1) $\boldsymbol{S}_i$ represents the spin angular momentum operator at the lattice site i and $J(>0)$, D and A denote the exchange, DM interaction and anisotropy parameters respectively. Here the easy axis of magnetization is chosen along the z-direction and the weak anisotropic axis corresponding to DM interaction is chosen parallel to $\boldsymbol{m} = (1, 1, 1)$. From the Hamiltonian (2.1), it may be observed that in the case of the weak or DM interaction a component of the spin at a given lattice site i interacts with the normal components of the nearest neighbours, unlike the exchange and crystal field anisotropic interactions. For large spin values, one can go to the classical limit, by replacing the spin angular momentum operator by a three-component spin vector $\boldsymbol{S}_i = (S_i^x, S_i^y, S_i^z)$. Now, the dynamics can be represented by the classical equation of motion $\mathrm{d}\boldsymbol{S}_n/\mathrm{d}t = \{\boldsymbol{S}_n, H\}_{\mathrm{PB}}$, where the Poisson bracket (PB) for two arbitrary functions f and g of spins can be written as [12]

$$\{f, g\}_{\mathrm{PB}} = \sum_{n=1}^{N} \sum_{\alpha,\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \frac{\partial f}{\partial S_n^\alpha} \frac{\partial g}{\partial S_n^\beta} S_n^\gamma. \tag{2.2}$$

Here $\epsilon_{\alpha\beta\gamma}$ is the Levi–Civita tensor. The lattice (discrete) equation of motion for the Hamiltonian (2.1) can now be written as

$$\frac{\mathrm{d}\boldsymbol{S}_n}{\mathrm{d}t} = \boldsymbol{S}_n \wedge [J(\boldsymbol{S}_{n+1} + \boldsymbol{S}_{n-1}) - D\boldsymbol{m} \wedge (\boldsymbol{S}_{n+1} - \boldsymbol{S}_{n-1}) - 2AS_n^z \boldsymbol{l}] \qquad \boldsymbol{S}_n^2 = 1, \boldsymbol{l} = (0, 0, 1). \tag{2.3}$$

In the long wavelength low temperature limit, we make the continuum approximation for the one-dimensional chain by introducing the Taylor expansion $\boldsymbol{S}_{n\mp1} = \boldsymbol{S}(x,t) \mp a\partial\boldsymbol{S}/\partial x + \frac{1}{2}a^2\partial^2\boldsymbol{S}/\partial x^2 + \cdots$ (where $x = na$, a is the lattice parameter) in equation (2.3) and after suitable rescaling of time and redefinition of the parameters D and A the resultant equation reads

$$\boldsymbol{S}_t = \boldsymbol{S} \wedge [\boldsymbol{S}_{xx} - D(\boldsymbol{m} \wedge \boldsymbol{S}_x) - 2AS^z\boldsymbol{l}] \qquad \boldsymbol{S}^2 = 1. \tag{2.4}$$

Here the subscripts represent partial derivatives. The form of equation (2.4), namely $\boldsymbol{S}_t = \boldsymbol{S} \wedge \boldsymbol{F}_{\mathrm{eff}}$, is commonly known as the Landau–Lifshitz (LL) equation, where the effective field $\boldsymbol{F}_{\mathrm{eff}} = \boldsymbol{S}_{xx} - D(\boldsymbol{m} \wedge \boldsymbol{S}_x) - 2AS^z\boldsymbol{l}$ is due to the magnetic interactions given in equation (2.1). Having derived the equation of motion the task now lies in solving the same to understand the underlying nonlinear dynamics. Experience shows that the equation of motion in the LL form is not convenient on many occasions for study of the nonlinear excitations. This difficulty is normally overcome by deriving suitable equivalent representations. Among them, the differential geometric approach (in other words the space curve formalism) for isotropic ferromagnets [13] and the stereographic projection technique [14] for anisotropic systems are very useful. Hence, in the following sections we study the dynamics of the isotropic and anisotropic systems separately after deriving the equivalent representations.

3. Dynamics of a weak isotropic ferromagnetic spin chain

We treat the dynamics of the weak isotropic ferromagnetic spin chain (WIFSC) by considering equation (2.4) (with $A = 0$) after deriving its equivalent representation through the space curve formalism [13]. For this purpose we identify the state of the weak ferromagnetic spin chain with that of a moving helical space curve in E^3. A local coordinate system $e_i (i = 1, 2, 3)$ on the space curve is formed by mapping the spin vector $S(x, t)$ with the unit tangent vector $e_1(x, t)$ and defining the unit normal vectors $e_2(x, t)$ and $e_3(x, t)$ in the usual way. The change of orientation of the trihedron e_i which defines the space curve uniquely within rigid motion is determined by the Serret–Frenet equations [15]

$$\begin{pmatrix} e_{1x} \\ e_{2x} \\ e_{3x} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{3.1}$$

Here $\kappa = (e_{1x} \cdot e_{1x})^{1/2}$ is the curvature and $\tau = \kappa^{-2} e_1 \cdot (e_{1x} \wedge e_{1xx})$ is the torsion of the space curve. In view of the above identifications, using equation (2.4) with $A = 0$ the evolution of the tangent vector can be written as

$$e_{1t} = e_1 \wedge [e_{1xx} - D(m \wedge e_{1x})]. \tag{3.2a}$$

Then the evolution of the trihedron $e_i, i = 1, 2, 3$ preserving their orthogonality is determined by equations (3.1) and (3.2a) and is written as

$$\begin{pmatrix} e_{1t} \\ e_{2t} \\ e_{3t} \end{pmatrix} = \begin{pmatrix} 0 & -\kappa\tau + D\kappa & \kappa_x \\ \kappa\tau - D\kappa & 0 & (\kappa_{xx}/\kappa) - \tau^2 + D\tau \\ -\kappa_x & -(\kappa_{xx}/\kappa) + \tau^2 - D\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{3.2b–d}$$

While deriving equations (3.2*b*–*d*), in the case of weak DM interaction, we considered only those contributions during spin evolution that occur due to small angle variation of spins with reference to the weak anisotropic axis (parallel to m). The conditions for compatibility of the Serret–Frenet equations (3.1) with equations (3.2*b*–*d*) for the evolution of the trihedron given by $(e_{ix})_t = (e_{it})_x$, $i = 1, 2, 3$ lead to the following evolution equations for the curvature and torsion of the space curve.

$$\kappa_t = -2\kappa_x \tau - \kappa \tau_x + D\kappa_x \tag{3.3a}$$

$$\tau_t = \kappa\kappa_x + [(\kappa_{xx}/\kappa) - \tau^2 + D\tau]_x. \tag{3.3b}$$

Equations (3.3) represent the dynamics of the weak isotropic ferromagnetic spin chain under the small angle variation mentioned above in an equivalent representation (related to the energy and current densities). In order to identify equations (3.3) with more standard nonlinear partial differential equations we make the complex transformation [3] $q = \frac{1}{2}\kappa e^{i\int \tau \, dx}$ and obtain

$$iq_t + q_{xx} + 2|q|^2 q - iDq_x = 0. \tag{3.4}$$

Equation (3.4), after making a Galilean-like transformation, reduces to the well known completely integrable nonlinear Schrödinger (NLS) equation,

$$iq_t + q_{xx} + 2|q|^2 q = 0 \tag{3.5}$$

possessing N-soliton solutions [16]. From the knowledge of the soliton solutions of equation (3.5), the N-soliton solutions for the energy and current densities of the spin chain which are related to the curvature and torsion of the space curve can be constructed. Also, it is a standard procedure in classical differential geometry that given the curvature and torsion, the space curve can be constructed uniquely within rigid motions [17] and hence the spins. For example, the final form of the one-soliton solution for the spin vector ($S = (S^x, S^y, S^z)$) can be written as [18]

$$S^x = \nu \operatorname{sech} \Psi[\eta_0 \tanh \Psi \sin \chi - \xi_0 \cos \chi] \quad (3.6a)$$

$$S^y = -\nu \operatorname{sech} \Psi[\eta_0 \tanh \Psi \cos \chi + \xi_0 \sin \chi] \quad (3.6b)$$

$$S^z = 1 - \nu\eta_0 \operatorname{sech}^2 \Psi. \quad (3.6c)$$

where

$$\nu = \frac{2\eta_0}{(\xi_0^2 + \eta_0^2)} \qquad \Psi = (x - 2\xi_0 t - \theta_0) \qquad \chi = \xi_0(x - \theta_0). \quad (3.6d)$$

Here η and ξ are associated with the eigenvalue of the problem and θ_0 is a constant. So, we conclude that the WIFSC becomes integrable and the elementary spin excitations are governed by solitons when we consider the effective field contribution due to weak (DM) interaction lies only within a small angle cone whose axis lies parallel to $\boldsymbol{m}$.

4. Dynamics of a weak anisotropic ferromagnetic spin chain

Having analysed the dynamics of a WIFSC we now try to investigate that of an anisotropic chain (WAFSC) by considering equation (2.4) under stereographic projection. We stereographically project the unit sphere of spins onto a complex plane by defining [14]

$$\omega \equiv \omega_R + i\omega_I = \frac{S^x + iS^y}{1 + S^z} \quad (4.1)$$

ω_R, ω_I; real. Using equation (4.1) and their derivatives in the component equations of (2.4), after lengthy calculations we obtain

$$(1 + \omega\omega^*)(i\omega_t + \omega_{xx}) - 2\omega^*\omega_x^2 + 2A(1 - \omega\omega^*)\omega - iD\{2(\omega_R + \omega_I) + (1 - \omega\omega^*)\omega_x\} = 0. \quad (4.2)$$

The dynamics of the spin chain can be understood by solving the above equation.

4.1. Search for integrable models

Before actually solving equation (4.2), in order to see whether it is integrable in general and, if not, for any specific parametric choices of A and D, we carry out the Painlevé singularity structure analysis following the algorithmic procedure of Ablowitz *et al* [19], originally developed for ordinary differential equations and later on extended to partial differential equations by Weiss *et al* [20]. This method has been found to be successful in identifying a number of integrable ferromagnetic models [9, 21]. For this, we rewrite equation (4.2) and its complex conjugate equation by replacing ω and ω^* by F and H respectively and obtain

$$(1 + FH)(iF_t + F_{xx}) - 2HF_x^2 + 2A(1 - FH)F$$
$$-iD\{(F + H) - i(F - H) + (1 - FH)\}F_x = 0 \quad (4.3a)$$

$$(1 + FH)(-iH_t + H_{xx}) - 2FH_x^2 + 2A(1 - FH)H$$
$$+iD\{(F + H) - i(F - H) + (1 - FH)\}H_x = 0. \quad (4.3b)$$

In order to understand the singularity structure of the solutions to equations (4.3), we assume the solution in the form of a Laurent series given by

$$F = F_0(x, t)\phi^p(x, t) + \sum_{j=1} F_j(x, t)\phi^{p+j}(x, t) \qquad F_0 \neq 0 \quad (4.4a)$$

$$H = H_0(x, t)\phi^q(x, t) + \sum_{j=1} H_j(x, t)\phi^{q+j}(x, t) \qquad H_0 \neq 0 \quad (4.4b)$$

where $\phi(x,t)$ is a non-characteristic singular manifold. In order to reduce the complexity of calculations we follow Kruskal's ansatz (see, for example, [21]) by assuming $\phi(x,t) = x + \psi(t)$ ($\psi(t)$: arbitrary) in which case F_0, H_0 and F_j, H_j are functions of t alone. Using the leading order of solutions (4.4) ($F \sim F_0\phi^p$, $H \sim H_0\phi^q$) in equations (4.3) and balancing the dominant terms, we solve for p, q and F_0, H_0 and obtain three different cases of solutions. Then we substitute the generalized Laurent expansions $F = F_0\phi^p + \cdots + \alpha\phi^{p+r} + \cdots$ and $H = H_0\phi^q + \cdots + \beta\phi^{q+r} + \cdots$ (α, β arbitrary), in equations (4.3) containing dominant terms alone and find the resonance values (r) for all three cases of solutions obtained. The results are as follows

$$\text{Case (i): } p=-1, q=-1 \qquad r = 0, 0, -1, -1 \tag{4.5a}$$

$$\text{Case (ii): } p=-1, q=0 \qquad r = 0, 0, -1, 1 \tag{4.5b}$$

$$\text{Case (iii): } p=0, q=-1 \qquad r = 0, 0, -1, -1. \tag{4.5c}$$

To all three cases F_0 and H_0 are found to be arbitrary, which is in agreement with the resonance values $r = 0, 0$. The resonance $r = -1$ represents the arbitrariness of the singular manifold. Upon verification we further find that, in cases (ii) and (iii), F_1 (or H_1) *is arbitrary in accordance with the resonance value* $r = 1$.

There is one more possibility, with the leading powers $p = 0$, $q = 0$, which corresponds to the Taylor expansion of the regular solutions [21].

$$F = F_0(t) + F_1(t)\phi + F_2(t)\phi^2 + \cdots \tag{4.6a}$$

$$H = H_0(t) + H_1(t)\phi + H_2(t)\phi^2 + \cdots. \tag{4.6b}$$

We calculate the coefficients $F_0, F_1, \ldots$ and their conjugates $H_0, H_1, \ldots$ by substituting equations (4.6) into (4.3) and collecting the coefficients of different powers of ϕ. As there is no possibility of branching unless the coefficients of the highest derivative vanish, we assume in the coefficients of ϕ^0

$$(1 + F_0H_0) = 0 \tag{4.7a}$$

and hence obtain

$$-H_0F_1^2 + 2AF_0 - \mathrm{i}DF_1 - (\mathrm{i}DF_1/2)[(F_0 + H_0) - \mathrm{i}(F_0 - H_0)] = 0 \tag{4.7b}$$

and its conjugate equation. Equation (4.7a) indicates that either F_0 or H_0 is arbitrary. Collecting the coefficients of ϕ and on using equations (4.7) we obtain

$$F_2 = \frac{\mathrm{i}}{2}(F_{0t} + F_1\psi_t) - AF_0 + \frac{\mathrm{i}}{2}DF_1 - \frac{1}{(F_0H_1 + H_0F_1)} \times \left[F_1^2H_1 - 2AF_1 + \frac{\mathrm{i}D}{2}\{(1-\mathrm{i})F_1^2 + (1+\mathrm{i})F_1H_1\}\right] \tag{4.8}$$

and its conjugate equation for H_2. Finally, collecting the coefficients of ϕ^2 and on using equations (4.7)–(4.8) we find that the coefficients F_3 and H_3 vanish identically indicating that F_3 and H_3 are arbitrary, leaving the resonance condition which is satisfied identically only when $D = 0$. The above analysis shows that equation (4.2) possesses the Painlevé property and is expected to be integrable only when $D = 0$. Otherwise it indicates the presence of movable critical manifolds involving logarithmic type branches.

4.2. Soliton excitations

After finding that the anisotropic weak ferromagnet is not expected to be integrable in general, the natural question arises as to whether there exists any approximate integrable model of the weak anisotropic ferromagnet whose elementary spin excitations can be expressed in terms of solitons. We choose the direction of the weak anisotropic axis due to DM interaction parallel to the easy axis of magnetization (i.e. $\boldsymbol{m} = \boldsymbol{l} = (0, 0, 1)$, z-direction) and hence the LL equation (2.4) can be rewritten as

$$S_t = S \wedge [S_{xx} - D(l \wedge S_x) - 2AS^z l] \qquad S^2 = 1. \tag{4.9}$$

Now, we define

$$u = S^x + \mathrm{i}S^y \tag{4.10}$$

and in view of the constraint $S^2 = 1$, we then have

$$(S^z)^2 = 1 - |u|^2 \tag{4.11}$$

which for small deviations of the spin S from the equilibrium direction that lies parallel to the easy axis of magnetization (anisotropy axis) and also to the weak anisotropic axis, can be written as

$$S^z = 1 - \tfrac{1}{2}|u|^2. \tag{4.12}$$

Using equations (4.10) and (4.12) in the component equations of (4.9) and under the long wavelength approximation [1] by keeping only nonlinear terms of magnitude $|u|^2 u$, we obtain

$$\mathrm{i}u_t - u_{xx} + A|u|^2 u - 2Au - \mathrm{i}Du_x = 0. \tag{4.13}$$

Upon making a Galilean-type transformation and transforming u as $u \to (2/A)^{1/2} u \mathrm{e}^{-2\mathrm{i}At}$, equation (4.13) reduces to the integrable NLS equation

$$\mathrm{i}u_t - u_{xx} + 2|u|^2 u = 0. \tag{4.14}$$

The NLS equations (3.5) and (4.14) are also familiar in nonlinear optics [22]. It may be noted that, in optical terms, while equation (3.5) possess bright soliton solutions, equation (4.14) admits dark solitons., This is because of the change of sign of the dispersive term u_{xx} of both the equations. Knowing the N-soliton solutions for u, the corresponding solitons for the spin vectors can straight away be constructed using equations (4.10) and (4.11). For instance, the one soliton solution for the spin vectors can be written as [23]

$$S^x + \mathrm{i}S^y = \tfrac{1}{2}\alpha_1[(1 + g_1) - (1 - g_1)\tanh(\tfrac{1}{2}\xi_1)]\exp(\mathrm{i}\zeta_1) \tag{4.15a}$$

where

$$\xi_1 = k_1 x - \Omega_1 t + \xi_1^{(0)} \qquad \zeta_1 = l_1 x - (\lambda_1 - l_1^2)t + \zeta_1^{(0)}$$
$$g_1 = (-1/\beta_1)[k_1^2 - \mathrm{i}(\Omega_1 + 2l_1 k_1)]^2 \qquad |\alpha_1|^2 = (\beta_1/4k_1^2) \qquad \beta_1 = k_1^4 + (\Omega_1 + 2l_1 k_1)^2$$
$$\lambda = \text{constant}. \tag{4.15b}$$

Thus we find that in weak anisotropic ferromagnets when the weak anisotropic axis lies parallel to the easy axis of magnetization under the long wavelength approximation and for small angle deviation of the spins the dynamics is found to be governed by the completely integrable NLS equation and the elementary spin excitations by solitons.

4.3. A class of spin waves

Having identified an approximate integrable model of the weak anisotropic ferromagnet whose elementary excitations are governed by solitons, we now try to investigate the nature of spin waves in the more general case. For this, we rewrite equation (4.9) in terms of the stereographic variable using equation (4.1). The resultant equation takes the form

$$(1+\omega\omega^*)(\mathrm{i}\omega_t+\omega_{xx})-2\omega^*\omega_x^2+(1-\omega\omega^*)(2A\omega-\mathrm{i}D\omega_x)=0. \tag{4.16}$$

An obvious but interesting class of time-dependent solution to equation (4.16) is the plane wave solution

$$\omega=B\exp\mathrm{i}[\Omega t-kx+\mu] \tag{4.17}$$

with the dispersion relation

$$\Omega(1+B^2)+(k^2-Dk+2A)(1-B^2)=0. \tag{4.18}$$

Here B and μ are the amplitude and phase of the wave respectively. Using equation (4.17) in (4.1), the spin components can be written as

$$S^x=\frac{2B}{(1+B^2)}\cos(\Omega t-kx+\mu) \tag{4.19a}$$

$$S^y=\frac{2B}{(1+B_2^2)}\sin(\Omega t-kx+\mu) \tag{4.19b}$$

$$S^z=\frac{(1-B)}{(1+B^2)}. \tag{4.19c}$$

It may be noted that the spin wave oscillations (4.19) are confined to the S^x–S^y plane with constant energy density $\varepsilon=\frac{1}{2}k^2B^2$.

More general spin wave solutions can be obtained by rewriting equation (4.16) in terms of the wave variable $\zeta=\Omega t-kx$. After making the transformation

$$\omega=a(\zeta)\exp\left[\mathrm{i}\int b(\zeta)\,\mathrm{d}\zeta\right]\qquad \zeta=\Omega t-kx \tag{4.20}$$

equation (4.16) can be rewritten as a set of coupled equations:

$$-\Omega a_\zeta=k^2[2a_\zeta b+ab_\zeta]-\frac{4k^2a^2ba_\zeta}{(1+a^2)}+\frac{Dk(1-a^2)}{(1+a^2)}a_\zeta \tag{4.21a}$$

$$\Omega ab=k^2(a_{\zeta\zeta}-ab^2)-\frac{2k^2a}{(1+a^2)}(a_\zeta^2-a^2b^2)+\frac{(1-a^2)}{(1+a^2)}(2Aa-Dkab). \tag{4.21b}$$

Equation (4.21a) can be integrated to give

$$\frac{\Omega}{2(1+a^2)}=\frac{k^2a^2b}{(1+a^2)^2}+\frac{Dka^2}{2(1+a^2)^2}+\varepsilon_1 \tag{4.22}$$

where ε_1 is the integration constant. Making use of equation (4.22) in equation (4.21b) and integrating once, we get

$$a_\zeta^2=(1+a^2)^2\left\{\left[\varepsilon_2+\left(\varepsilon_1^2+\frac{\Omega}{4k^4}(\Omega+4\varepsilon_1k^2)\right)\frac{1}{a^2}+\varepsilon_1^2a^2\right]\right.$$
$$\left.+\left(\frac{2A}{k^2}+\frac{D^2}{4k^2}-\frac{D\Omega}{2k^3}\right)\frac{a^2}{(1+a^2)^2}-\left(\frac{D\Omega}{2k^3}\right)\frac{1}{(1+a^2)^2}\right\} \tag{4.23}$$

where ε_2 is the second integration constant. Choosing $\varepsilon_1 = \varepsilon_2 = 0$, equation (4.23) when $3D^2 = 8A$ can be rewritten as

$$a_\zeta^2 = \frac{1}{(4k^2a^2)}\left[\frac{\Omega}{k}(1+a^2) - (8A+D^2)^{1/2}a^2\right]^2. \tag{4.24}$$

On integrating equation (4.24), we obtain

$$a^2 = \lambda(\varepsilon_3 \exp(\delta\zeta) - 1) \tag{4.25}$$

where $\lambda = 1/(1-D)$, $\delta = (\Omega - 2Dk^2)/k^2$ and ε_3 is the new integration constant. From equation (4.22) (with $\varepsilon_1 = 0$) we can write

$$2b = \frac{\Omega}{k^2}\left[\frac{\varepsilon_3 \exp(\delta\zeta) + (\lambda - 1)}{\varepsilon_3 \exp(\delta\zeta) - 1}\right] - (D/k). \tag{4.26}$$

Knowing a and b, ω can be computed using equation (4.20). Finally, using equation (4.1) the spin components can be constructed as

$$S^+ \equiv S^x + \mathrm{i}S^y = \frac{\lambda^{-1/2}(\varepsilon_3 \exp(\delta\zeta) - 1)^{1/2}}{1 + \lambda(\varepsilon_3 \exp(\delta\zeta) - 1)} \exp(\mathrm{i}\Theta) \tag{4.27a}$$

$$S^z = \frac{(\lambda + 1) - \varepsilon_3 \exp(\delta\zeta)}{(\lambda - 1) + \varepsilon_3 \exp(\delta\zeta)} \tag{4.27b}$$

where

$$\Theta = \varepsilon_4 + \frac{\Omega}{2k^2}\left[\left(2 - \lambda - \frac{D}{2k}\right)\zeta - \ln[\exp(\zeta)(\varepsilon_3 \exp(\delta\zeta) - 1)^{-\lambda/\delta}]\right] \tag{4.27c}$$

and ε_4 is a constant of integration.

5. Conclusions

In this paper, we have discussed in detail the nonlinear spin dynamics of both isotropic and anisotropic classical one-dimensional continuum Heisenberg ferromagnetic spin chains with the Dzialoshinski–Moriya weak interaction. After constructing the Landau–Lifshitz equation to represent the spin dynamics we analysed both isotropic and anisotropic weak ferromagnets under the space curve and stereographic representations respectively. The weak isotropic system is found to be integrable and the elementary spin excitations governed by solitons, when the effective field due to weak interaction is considered within a small angle (between s and m) cone. A Painlevé singularity structure analysis of the weak anisotropic system showed that the system is, in general, not expected to be integrable. However, under the long wavelength approximation and for small angle variation of spins, the anisotropic system when the weak anisotropic axis lies parallel to the easy axis of magnetization was found to be integrable and the elementary spin excitations governed by solitons. We have also reported a class of spin wave solutions to represent the dynamics of the weak anisotropic ferromagnets in the more general case. The arbitrary general case which is not integrable will be taken up separately in the future for the possible study of chaotic behaviour and will be reported elsewhere.

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